

Thermoelectric characteristics of \mathbb{Z}_k parafermion Coulomb islands

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Abstract Using the explicit rational conformal field theory partition functions for the \mathbb{Z}_k parafermion quantum Hall states on a disk we compute numerically the thermoelectric power factor for Coulomb-blockaded islands at finite temperature. We demonstrate that the power factor is rather sensitive to the neutral degrees of freedom and could eventually be used to distinguish experimentally between different quantum Hall states having identical electric properties. This might help us to confirm whether non-Abelian quasiparticles, such as the Fibonacci anyons, are indeed present in the experimentally observed quantum Hall states.

1 Introduction: non-Abelian anyons and Topological Quantum Computation

We shall start this section with the question of what non-Abelian statistics is. It is well known that when we exchange indistinguishable particles the quantum state acquires a phase $e^{i\pi(\theta/\pi)}$ which is proportional to the statistical angle θ/π . In three-dimensional coordinate space this phase can be either 0 when the particles are bosons, or 1 when the particles are fermions. In two-dimensional space, however, this restriction is not valid and the particles can have any statistical angle between 0 and 1, that's why they are called anyons. For example, the Laughlin anyons corresponding to the fractional quantum Hall (FQH) state with filling factor $1/3$ have $\theta_L/\pi = 1/3$. In addition, while the n -particle quantum states in three dimensions are constructed as representations of the symmetric group \mathcal{S}_n , which are symmetric for bosons and antisymmetric for fermions, in two dimensions the n -particle states are build up as representations of the braid group \mathcal{B}_n . The *non-Abelian anyons* are such particles in two dimensional space whose n -particle states belong to representations

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of \mathcal{B}_n whose dimension is bigger than 1. In terms of n -particle states this means that the non-Abelian anyons' wave functions belong to degenerate multiplets and that the statistical angle θ may be a non-trivial matrix, in which case the statistical phase $e^{i\theta}$ would be non-Abelian.

The anyonic states of matter are labeled by fusion paths [1] which are defined as concatenation of fusion channels and can be displayed in Bratteli diagrams. The fusion channels are denoted by the index 'c' in the fusion process of two particles of type a and b which is denoted symbolically as $\Psi_a \times \Psi_b = \sum_{c=1}^g N_{ab}^c \Psi_c$, where the fusion coefficients $(N_{ab})^c$ are integers which are symmetric and associative [2]. Put another way, a collection of particles $\{\Psi_a\}$ are non-Abelian anyons if $N_{ab}^c \neq 0$ for more than one c . As an example we consider the Ising anyons $\Psi_I(z) = \sigma(z) : e^{i\frac{1}{2\sqrt{2}}\phi(z)} :$ realized in a conformal field theory (CFT) with $\widehat{u(1)} \times$ Ising symmetry, where $\phi(z)$ is a normalized $\widehat{u(1)}$ boson and σ is the chiral spin field in the Ising CFT model,

$$\sigma \times \sigma = \mathbb{I} + \psi.$$

Besides the fact that non-Abelian statistics is a new fundamental concept in particle physics it is also important for the so-called topological quantum computation (TQC) [3, 4]. In this context quantum information is encoded into the fusion channels

$$\begin{aligned} |0\rangle &= (\sigma, \sigma)_{\mathbb{I}} & \longleftrightarrow & \sigma \times \sigma \rightarrow \mathbb{I} \\ |1\rangle &= (\sigma, \sigma)_{\psi} & \longleftrightarrow & \sigma \times \sigma \rightarrow \psi, \end{aligned}$$

which is a topological quantity—it is independent of the fusion process details depending only on the topology of the coordinate space. Fusion channel is also independent of the anyon separation and is preserved when the two particles are separated—if we fuse two particles and then split them again, their fusion channel does not change.

The basic idea of TQC is that quantum information can be encoded into the fusion channels and the quantum gates can be implemented by braiding non-Abelian anyons. As an illustration we can consider 8 Ising anyons, which in the quantum information language can be used to encode 3 topological qubits, and transport adiabatically anyon number 7 along a complete loop around anyon number 6. Then the 8-anyons states are multiplied by a statistical phase $(B_6^{(8,+)})^2 = X_3$ which implements the NOT gate $X_3 = \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes X$ on the third qubit [5, 3].

Another promising example of non-Abelian anyons are the Fibonacci anyons [6] realized in the diagonal coset of the \mathbb{Z}_3 parafermion FQH states [7, 8] (or, in the three-state Potts model) as the parafermion primary field ε corresponding to the nontrivial orbit of the simple-current's action $\mathbb{I} = \{\Lambda_0 + \Lambda_0, \Lambda_1 + \Lambda_1, \Lambda_2 + \Lambda_2\}$ $\varepsilon = \{\Lambda_0 + \Lambda_1, \Lambda_1 + \Lambda_2, \Lambda_0 + \Lambda_2\}$ with fusion rules

$$\mathbb{I} \times \mathbb{I} = \mathbb{I}, \quad \mathbb{I} \times \varepsilon = \varepsilon, \quad \varepsilon \times \varepsilon = \mathbb{I} + \varepsilon.$$

The information encoding for Fibonacci anyons is again in the fusion channels, denoted by the field \mathbb{I} and ε of the resulting fusion, however, this time for triples of anyons [6]

$$\begin{aligned} |0\rangle &= ((\varepsilon, \varepsilon)_{\mathbb{I}}, \varepsilon)_{\varepsilon} \\ |1\rangle &= ((\varepsilon, \varepsilon)_{\varepsilon}, \varepsilon)_{\varepsilon}, \end{aligned}$$

and the third state $((\varepsilon, \varepsilon)_{\varepsilon}, \varepsilon)_{\mathbb{I}}$, having a trivial quantum dimension, decouples from the previous two and is called non-computational [6].

Given that non-Abelian statistics is a new concept a natural question arises: how can it be discovered? In the rest of this paper we will discuss, how non-Abelian statistics might be observed in Coulomb-blockade conductance spectrometry and by measuring certain thermoelectric characteristics of Coulomb-blockaded islands.

2 Coulomb island spectroscopy

Let us consider a Coulomb-blockaded island, which can be realized as a quantum dot with drain, source and a side gate which is equivalent to single-electron transistor, like in Ref. [9]. This setup is an almost closed quantum system, which still has discrete energy levels and is like a large artificial atom but is highly tunable by Aharonov–Bohm flux and side-gate voltage.

2.1 Coulomb island's conductance–CFT approach

In this section we are going to use the chiral Grand canonical partition function for a disk fractional quantum Hall sample to calculate its thermoelectric properties. In such samples the bulk is inert due to the nonzero mobility gap while the edge is mobile and can be described by a rational unitary CFT [10, 11, 2]. The Grand partition function is

$$Z_{\text{disk}}(\tau, \zeta) = \text{tr}_{\mathcal{H}_{\text{edge}}} e^{-\beta(H - \mu N)} = \text{tr}_{\mathcal{H}_{\text{edge}}} e^{2\pi i \tau (L_0 - c/24)} e^{2\pi i \zeta J_0}, \quad (1)$$

where $H = \hbar \frac{2\pi \nu_F}{L} (L_0 - \frac{c}{24})$ is the edge Hamiltonian expressed in terms of the zero mode L_0 of the Virasoro stress-energy tensor (with a central charge c), $N = -\sqrt{\nu_H} J_0$ is the particle number on the edge expressed in terms of the zero J_0 mode of the $u(1)$ current and ν_H is the FQH filling factor. The trace is taken over the edge-states' Hilbert space $\mathcal{H}_{\text{edge}}$ which depends on the number of quasiparticles localized in the bulk. The temperature T and chemical potential μ are related to the modular parameters [2] τ and ζ by

$$\tau = i\pi \frac{T_0}{T}, \quad T_0 = \frac{\hbar v_F}{\pi k_B L}, \quad \zeta = i \frac{\mu}{2\pi k_B T},$$

where v_F is the Fermi velocity at the edge and L is the circumference of the edge. The CFT disk partition function in presence of AB flux $\phi = eB.A/h$, threading the disk, is modified by simply shifting the chemical potential [12]

$$\zeta \rightarrow \zeta + \phi \tau, \quad Z_{\text{disk}}^\phi(\tau, \zeta) = Z_{\text{disk}}(\tau, \zeta + \phi \tau).$$

It is interesting to note that the side-gate voltage V_g affects the quantum dot (QD) in the same way as the AB flux [13], through the externally induced electric charge [14] on the QD (continuous) $-C_g V_g/e \equiv v_H \phi = Q_{\text{ext}}$, where C_g is the capacitance of the gate.

The thermodynamic Grand potential on the edge is defined as usual as $\Omega(T, \mu) = -k_B T \ln Z_{\text{disk}}(\tau, \zeta)$ and the electron number can be computed as [15]

$$\langle N_{\text{el}}(\phi) \rangle_{\beta, \mu_N} = v_H \left(\phi + \frac{\mu_N}{\Delta \epsilon} \right) + \frac{1}{2\pi^2} \left(\frac{T}{T_0} \right) \frac{\partial}{\partial \phi} \ln Z_\phi(T, \mu_N) \quad (2)$$

Similarly the edge conductance in the linear-response regime can be computed by [15]

$$G(\phi) = \frac{e^2}{h} \left(v_H + \frac{1}{2\pi^2} \left(\frac{T}{T_0} \right) \frac{\partial^2}{\partial \phi^2} \ln Z_\phi(T, 0) \right). \quad (3)$$

There are certain difficulties in measuring QD conductance and distinguishing FQH states: experiments are performed in extreme conditions (high B , very low T) with expensive samples. Moreover, there are many doppelgangers [16], i.e., distinct states with the same conductance patterns with differences in the neutral sector where G not sensitive. Under these conditions the sequential tunneling of electrons one-by-one is dominating the cotunneling, which is a higher-order process associated with almost simultaneous virtual tunneling of pairs of electrons [14], that will not be considered here.

3 Thermopower: a finer spectroscopic tool

The thermopower, or the Seebeck coefficient, is defined [14, 13] as the potential difference V between the leads of the SET when $\Delta T = T_R - T_L \ll T_L$, under the condition that $I = 0$. Usually thermopower is expressed as $S = G_T/G$, where G and G_T are electric and thermal conductances, respectively. However, for the SET configuration $G_T \rightarrow 0$ and $G \rightarrow 0$ in the Coulomb blockade valleys, so that it is more appropriate to use another expression [14]

$$S \equiv - \lim_{\Delta T \rightarrow 0} \frac{V}{\Delta T} \Big|_{I=0} = - \frac{\langle \epsilon \rangle}{eT},$$

where $\langle \varepsilon \rangle$ is the average energy of the tunneling electrons. In the CFT approach using the Grand partition function for the FQH edge of the QD we can express the average tunneling energy as the difference between the energy of the QD with $N+1$ electrons and that of the QD with N electrons

$$\langle \varepsilon \rangle_{\beta, \mu_N}^\phi = \frac{E_{\text{QD}}^{\beta, \mu_{N+1}}(\phi) - E_{\text{QD}}^{\beta, \mu_N}(\phi)}{\langle N(\phi) \rangle_{\beta, \mu_{N+1}} - \langle N(\phi) \rangle_{\beta, \mu_N}}$$

The total QD energy (in the Grand canonical ensemble) can be written as

$$E_{\text{QD}}^{\beta, \mu_N}(\phi) = \sum_{i=1}^{N_0} E_i + \langle H_{\text{CFT}}(\phi) \rangle_{\beta, \mu_N},$$

where E_i , $i = 1, \dots, N_0$ are the occupied single-electron states in the bulk of the QD, and $\langle \dots \rangle_{\beta, \mu}$ is the Grand canonical average of H_{CFT} on the edge at inverse temperature $\beta = (k_B T)^{-1}$ and chemical potential μ . The chemical potentials μ_N and μ_{N+1} of the QD with N and $N+1$ electrons can be chosen as [13]

$$\mu_N = -\frac{1}{2}\Delta\varepsilon, \quad \mu_{N+1} = \frac{1}{2}\Delta\varepsilon, \quad \Delta\varepsilon = \hbar \frac{2\pi v_F}{L}.$$

Another important observable is the thermoelectric power factor[13] \mathcal{P}_T which is defined as the electric power P generated by the temperature difference ΔT

$$P = V^2/R = \mathcal{P}_T(\Delta T)^2, \quad \mathcal{P}_T = S^2 G, \quad (4)$$

where $R = 1/G$ is the electric resistance of the CB island. The average tunneling energy can be expressed in terms of the CFT averages of the Hamiltonian and particle number as follows [13]

$$\langle \varepsilon \rangle_{\beta, \mu_N}^\phi = \frac{\langle H_{\text{CFT}}(\phi) \rangle_{\beta, \mu_{N+1}} - \langle H_{\text{CFT}}(\phi) \rangle_{\beta, \mu_N}}{\langle N(\phi) \rangle_{\beta, \mu_{N+1}} - \langle N(\phi) \rangle_{\beta, \mu_N}} \quad (5)$$

The electron number average can be computed from Eq. (2) and the edge energy average can be obtained from the Grand potential $\Omega_\phi(T, \mu_N) = -k_B T \ln Z_\phi(T, \mu)$ in presence of AB flux ϕ as

$$\langle H_{\text{CFT}}(\phi) \rangle_{\beta, \mu_N} = \Omega_\phi(T, \mu_N) - T \frac{\partial \Omega_\phi(T, \mu_N)}{\partial T} - \mu_N \frac{\partial \Omega_\phi(T, \mu_N)}{\partial \mu}. \quad (6)$$

4 \mathbb{Z}_k parafermion quantum Hall islands

The CFT for the \mathbb{Z}_k parafermion quantum Hall islands (or QDs) contains an electric charge part $\widehat{u(1)}$ and a neutral part which is realized as a parafermion diagonal coset

[7]

$$\left(\widehat{u(1)} \otimes \frac{\widehat{su(k)_1} \oplus \widehat{su(k)_1}}{\widehat{su(k)_2}} \right)^{\mathbb{Z}_k}$$

The total disk partition function for the \mathbb{Z}_k parafermion quantum Hall islands [7] is labeled by two integers $l \bmod k+2$ and $\rho \bmod k$ satisfying $l - \rho \leq \rho \bmod k$ and can be written as follows

$$\chi_{l,\rho}(\tau, \zeta) = \sum_{s=0}^{k-1} K_{l+s(k+2)}(\tau, k\zeta; k(k+2)) \text{ch}(\Lambda_{l-\rho+s} + \Lambda_{\rho+s})(\tau), \quad (7)$$

where $K_l(\tau, k\zeta; k(k+2))$ are the chiral partition functions of the $\widehat{u(1)}$ part while $\text{ch}(\Lambda_\mu + \Lambda_\rho)(\tau)$ are the characters of the neutral part of the CFT. The $\widehat{u(1)}$ part corresponds to Luttinger liquid partition function (with compactification radius $R_c = 1/m$)

$$K_l(\tau, \zeta; m) = \frac{\text{CZ}}{\eta(\tau)} \sum_{n=-\infty}^{\infty} q^{\frac{m}{2}(n+\frac{l}{m})^2} e^{2\pi i \zeta (n+\frac{l}{m})}$$

where the modular parameter is related to the temperature

$$q = e^{-\beta \Delta \varepsilon} = e^{2\pi i \tau}, \quad \Delta \varepsilon = \hbar \frac{2\pi v_F}{L}$$

and the Dedekind function and Cappelli–Zemba factors [11, 2] are given by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \text{CZ} = e^{-\pi v_H \frac{(\text{Im} \zeta)^2}{\text{Im} \tau}}$$

The neutral partition function are labeled by a level-2 weight $\Lambda_\mu + \Lambda_\rho$ with the condition $0 \leq \mu \leq \rho \leq k-1$ and have the form [7]

$$\text{ch}_{\sigma, Q}(\tau) = q^{\Delta(\sigma) - \frac{c}{24}} \sum_{\substack{m_1, m_2, \dots, m_{k-1}=0 \\ \sum_{i=1}^{k-1} i m_i \equiv Q \bmod k}}^{\infty} \frac{q^{\underline{m} \cdot C^{-1} \cdot (\underline{m} - \Lambda_\sigma)}}{(q)_{m_1} \cdots (q)_{m_{k-1}}},$$

$$(q)_n = \prod_{j=1}^n (1 - q^j), \quad \Delta(\sigma) = \frac{\sigma(k - \sigma)}{2k(k+2)}, \quad c = \frac{2(k-1)}{k+2}$$

where $\underline{m} = (m_1, \dots, m_{k-1})$, $0 \leq \sigma \leq Q \leq k-1$ and C^{-1} is the inverse Cartan matrix for $su(k)$. The coset weight labels are related to σ and Q by $\mu = Q - \sigma$, $\rho = Q$. Using the explicit formulas (5) for the thermopower in terms of the average tunneling energy expressed in terms of the Grand canonical averages (2) and (6) and the partition function (7), as well as Eq. (3) for the conductance, we can compute the thermopower for the \mathbb{Z}_3 parafermion FQH state. We plot in Fig. 1 the electric conductance and thermopower as functions of the AB flux ϕ , or, equivalently

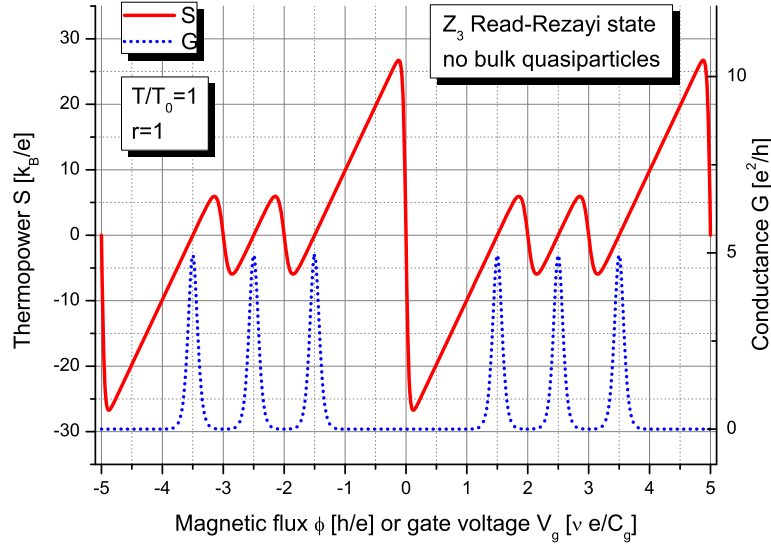


Fig. 1 Conductance peaks (right Y scale) and thermopower (left Y scale) for the \mathbb{Z}_3 parafermion FQH state without bulk quasiparticles.

as functions of the side-gate voltage V_g for the \mathbb{Z}_3 parafermion FQH state without quasiparticles in the bulk, i.e., for $l = 0$ and $\rho = 0$. Just like the thermopower of metallic quantum dots [14], we see in Fig. 1 that the peaks of the conductance precisely corresponds to the (continuous in the limit $T \rightarrow 0$) zeros of the thermopower.

Similarly, we can compute from Eq. (4) the power factor \mathcal{P}_T for the \mathbb{Z}_3 parafermion FQH state without quasiparticles in the bulk ($l = 0$ and $\rho = 0$), which is plotted together with the conductance in Fig. 2.

5 Conclusion and perspectives

The thermoelectric characteristics of Coulomb blockaded QDs, such as the thermopower and especially the thermoelectric power factor, appear to be more sensitive to the neutral modes in the FQH liquid than the tunneling conductance. These could be used as experimental signatures to identify (non-Abelian) Fibonacci anyons[6], which are believed to exist in the $\nu_H = 12/5$ quantum Hall state. This experimentally observed FQH state [17] might be a realization of the particle-hole conjugate of the \mathbb{Z}_3 parafermion quantum Hall state in the second Landau level with filling factor $\nu_H = 3 - k/(k+2)$ for $k = 3$.

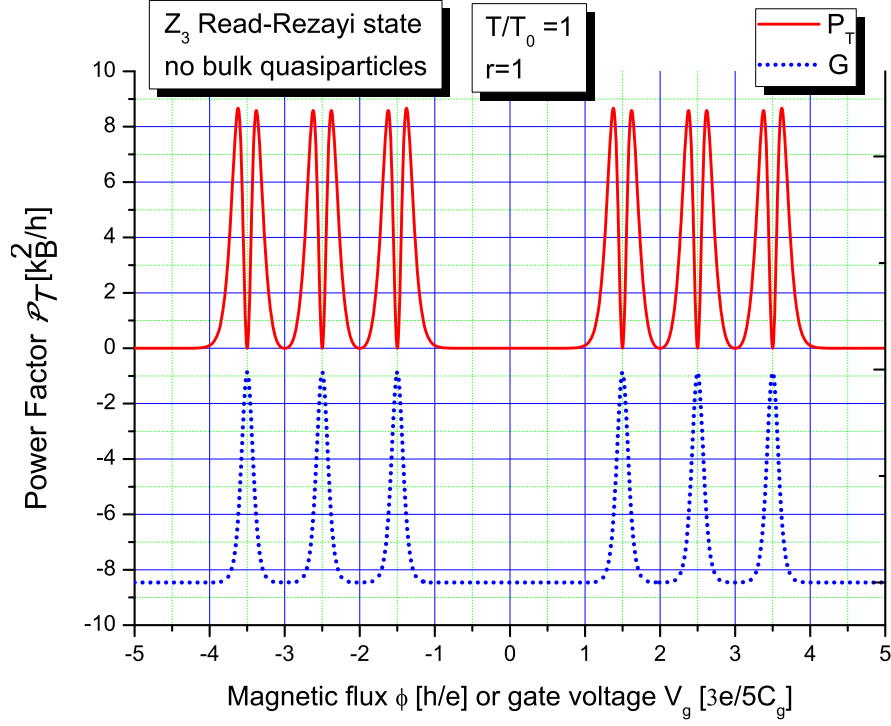


Fig. 2 Power factor and conductance for the \mathbb{Z}_3 parafermion FQH state without bulk quasiparticles.

A recent experiment [18] demonstrated that the power factor of a Coulomb blockaded quantum dot might be directly measurable like the observable plotted in Fig. 3c there. This could allow us to estimate from the experiment the ratio between the Fermi velocities of the charged and neutral edge modes by comparing with the power factor profile computed theoretically from the CFT [8]. Finally, this possibility to distinguish neutral characteristics of FQH states is bringing a new hope that we could eventually decide whether Fibonacci anyons are indeed realized in Nature.

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